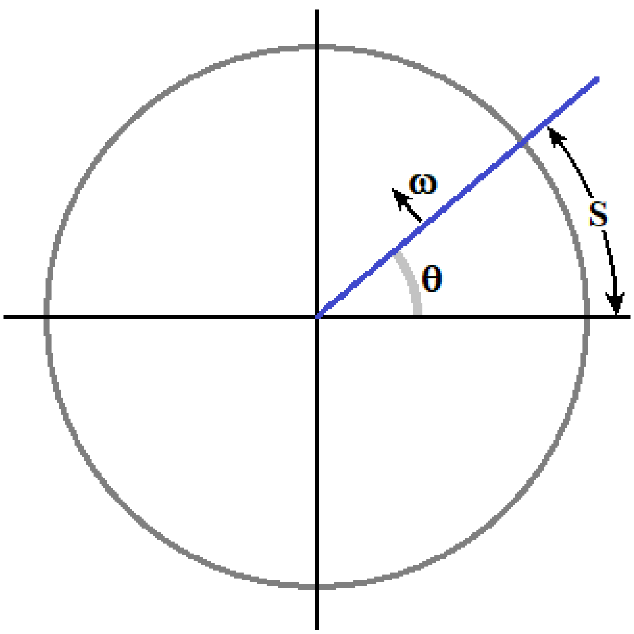
Part 9: Rotational Motion

University Physics V1 (Openstax): Chapter 10 & 11
Physics for Engineers & Scientists (Giancoli): Chapter 8

Rotational Kinematics

For rotating objects, velocity is not a universal variable. Different parts of the object move at different speeds. Distance (x) is also not a universal variable. Consequently, these are not good variables to describe rotational motion.

- Angular Position (θ) fills the role of x.
- Initial Angular Position (θ_0) is the angular position at t=0, and it fills the role of x_0.
- Angular Displacement ($\Delta\theta = \theta - \theta_0$) fills the role of Δx.
- Arc Length (S) is the distance a part of the rigid object moves.



1 rotation/revolution = 2π radians = 360°

$$S = r \cdot \Delta\theta \; (radians)$$

$$\theta = \frac{S}{r} \qquad \frac{meters}{meters} = no\ units$$

Note: 'Radians' is a 'dummy' unit.

- Angular Velocity (ω) fills the role of v. (*ω is also called 'Angular Frequency'*)
- Initial Angular Velocity (ω_0) is the angular velocity at t=0, and it fills the role of v_0.

Note: 'ω' is a lower-cased Greek letter omega. Not W.

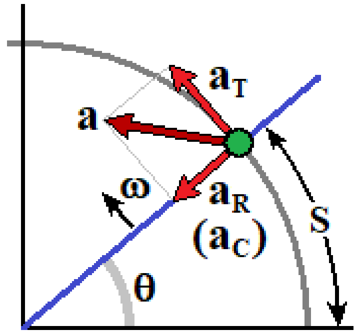
$$\omega_{Avg} = \frac{\Delta\theta}{\Delta t} = \frac{\theta - \theta_0}{t} \qquad \omega = \frac{d\theta}{dt} \qquad \text{Units: } \frac{rad}{s} \qquad r\omega = r\frac{d\theta}{dt} = \frac{d(r\theta)}{dt} = \frac{dS}{dt} = v$$

- The Period (T) is the time needed to make one full revolution.
- The Frequency (f) is the rotation rate (number of revolutions per unit time)

$$\text{Units: } 1\,\frac{\text{Revolution}}{\text{Second}} = 1\text{ s}^{-1} = 1\text{ Hz} \qquad 60\text{ RPM} = 60\,\frac{\text{Revolution}}{\text{Minute}} = 1\text{ Hz}$$

$$f = \frac{1}{T} \qquad \omega = 2\pi f = \frac{2\pi}{T} \qquad v = \frac{2\pi r}{T} = \omega r$$

- In rotational motion, Acceleration (a) gets broken down into two components.



- Tangential Accleration (a_T) changes the speed (magnitude of velocity) of an object moving in a circle.
- Radial Accleration (a_R), equivalent to centripetal acceleration (a_C), changes the direction but not the speed of an object moving in a circle.

$$a = \sqrt{a_T^2 + a_R^2}$$

- Angular Acceleration (α) fills the role of a.

$$\alpha_{Avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t} \qquad \alpha = \frac{d\omega}{dt} \qquad \text{Units: } \frac{rad}{s^2}$$

$$r\alpha = r\frac{d\omega}{dt} = \frac{d(r\omega)}{dt} = \frac{dv}{dt} = a_T \qquad a_R = a_C = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

- For the special case of constant angular acceleration, a set of equations can be found from the one-dimensional kinematic equations for constant acceleration.

$$v = v_0 + a_T t \qquad \omega r = \omega_0 r + \alpha r t \qquad \omega = \omega_0 + \alpha t$$

$$s = s_0 + \frac{1}{2}(v + v_0)t \qquad r\theta = r\theta_0 + \frac{1}{2}(r\omega + r\omega_0)t \qquad \theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t$$

$$s = s_0 + v_0 t + \frac{1}{2}a_T t^2 \qquad r\theta = r\theta_0 + r\omega_0 t + \frac{1}{2}r\alpha t^2 \qquad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v^2 = v_0^2 + 2a_T(s - s_0) \qquad r^2\omega^2 = r^2\omega_0^2 + 2r\alpha(r\theta - r\theta_0) \qquad \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

  - Rotational kinematics are similar to one-dimensional kinematics (just a change of variable) and solved the same way.
  - There are four variables (θ, ω, α, and t) and two constants (θ_0 and ω_0).
  - Three of the variables are related in each of the four equations. In many cases you can find the equation you need by determining which variable is absent.

Example: A wind turbine is activated as the winds reach a threshold. The blades start from rest and accelerate uniformly to an angular velocity of 9.87 rpms in 27.4 s. Determine the angular acceleration of the blades.

Extract Data: $\omega_0 = 0$ ω = 9.87 rpms = 1.03358 rad/s t = 27.4 s α = ???

$$\omega = 9.87\left(\frac{Rev}{Min}\right)\left(\frac{1\ Min}{60\ s}\right)\left(\frac{2\pi\ rad}{1\ Rev}\right) = 1.03358\frac{rad}{s}$$

Be warned: 2π/60 = 0.10472. If you fail to do this conversion, you will be off by a factor that is close to a power of 10.

No information about position is given. The equation without position is…

$$\omega = \omega_0 + \alpha t = \alpha t \qquad \alpha = \frac{\omega}{t} = \frac{1.03358\frac{rad}{s}}{27.4\ s} = 0.03772\frac{rad}{s^2}$$

Example: A grinding wheel undergoes uniform angular acceleration from rest to 680 rad/s over 1.30 seconds. Then the power is removed and friction causes it to decelerate back to rest in 18.7 seconds. Through what angle does the wheel turn during this time?

There are two different accelerations (both constants). This requires two sets of equations, one for the acceleration and one for the deceleration. This is an odd case where both are the same.

Accelerating:

Extract Data: $\theta_0 = 0$ $\theta = ???$ $\omega_0 = 0$ $\omega = 680\frac{rad}{s}$ $\alpha =$ $t = 1.30\,s$

Equation with no α: $\theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t$ $\theta = \frac{1}{2}\omega t = \frac{1}{2}\left(680\frac{rad}{s}\right)(1.30\,s) = 442\,rad$

Decelerating:

Extract Data: $\theta_0 = 442\,rad$ $\theta = ???$ $\omega_0 = 680\frac{rad}{s}$ $\omega = 0$ $\alpha =$ $t = 18.7\,s$

Equation with no α: $\theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t$

$$\theta = \theta_0 + \frac{1}{2}\omega_0 t = 442\,rad + \frac{1}{2}\left(680\frac{rad}{s}\right)(18.7\,s) = 6800\,rad$$

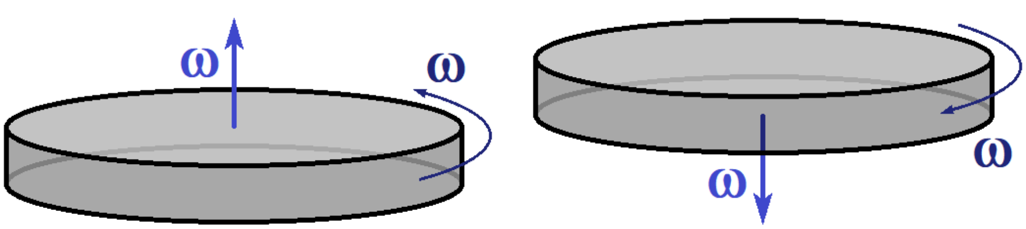
Alternatively, one could note that the average angular velocity is the same value whether accelerating or decelerating.

The solution would just be $\omega_{avg}t_{net}$

$$\omega_{avg} = \frac{1}{2}\omega = \frac{1}{2}\left(680\frac{rad}{s}\right) = 340\frac{rad}{s} \qquad \theta = \omega_{avg}t = \left(340\frac{rad}{s}\right)(1.30\,s + 18.7\,s) = 6800\,rad$$

Rotational Vectors

- While we are able to treat rotational variables one-dimensionally in most cases, they are still vectors with magnitude and direction.
- The direction of rotational vectors is defined to be either parallel or anti-parallel to the axis of rotation. Anti-parallel means parallel but pointing the opposite direction.
- If the object is rotating counter-clockwise in an xy-plane when viewed from above, then ω points upward in the z-direction.
- If the object is rotating clockwise in an xy-plane when viewed from above, then ω points downward in the negative z-direction.



Rotational Dynamics

- As rotational kinematics showed many similarities with one-dimensional translational kinematics, we can expect more similarities to appear in dynamics, but there are some differences too.

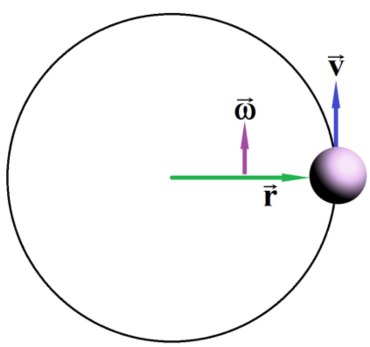
Concept	Translational (1D)	Rotational	Relationship
Position	x (or S)	θ	S = rθ
Velocity	v	ω	v = rω
Acceleration	a	α	$a_T = r\alpha$ $a_R = \omega^2 r$
Cause of Acceleration	F	τ (Torque)	$\vec{\tau} = \vec{r} \times \vec{F}$ $(\lvert\tau\rvert = rF\sin\theta)$
Inertia	m	I (Moment of Inertia)	$dI = r^2 dm$
Newton's 2nd Law	$\vec{F} = m\vec{a}$	$\vec{\tau} = I\vec{\alpha}$	
Work	W = Fd	W = τθ	
Kinetic Energy	$KE_{Translational} = ½mv^2$	$KE_{Rotational} = ½I\omega^2$	
Momentum	$\vec{P} = m\vec{v}$	$\vec{L} = I\vec{\omega}$	$\vec{L} = \vec{r} \times \vec{P}$
Force/Momentum	$\vec{F} = \frac{d\vec{P}}{dt}$	$\vec{\tau} = \frac{d\vec{L}}{dt}$	

Moment of Inertia (I)

- The moment of inertia is the rotational equivalent of mass (a resistance to being spun).

An object in uniform circular motion can be looked at as either rotational motion or as in linear translational motion (at least temporarily).

The kinetic energy of this motion should be the same either way it is calculated. This allows us to determine a relationship between moment of Inertia (I) and mass (m).



$$KE_{Translational} = \frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2 = \frac{1}{2}mr^2\omega^2$$

$$KE_{Rotational} = \frac{1}{2}I\omega^2 \qquad \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2$$

$$I = mr^2$$

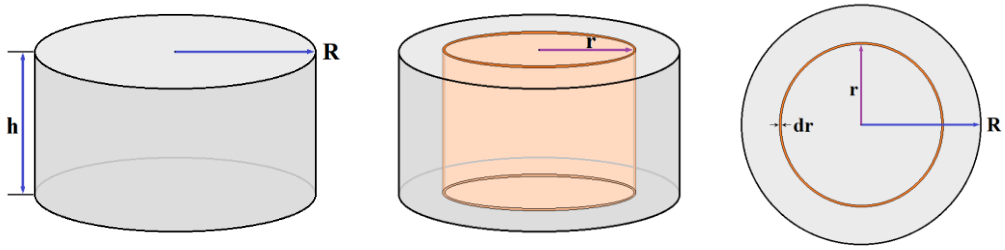
Note: This is only valid when all the mass is located at the same distance (r) from the axis of rotation.

- When the mass is not all at the same value of r, you must calculate the mass at each value of r and sum them all up. As there is often a continuous distribution of r values, summing may mean integration.

$$dI = r^2 dm \qquad I = \int_{r_1}^{r_2} r^2 dm \qquad dm = \rho dV$$

Example: Determine the moment of inertia of a solid cylindrical disk of uniform density, mass M, and radius R.

Step 1: Split mass into small pieces, each with the same value of r (as r appears in our equation). In this case, it will be infinitesimally thin cylindrical shells.



The left image shows the cylinder and its dimensions (R and h).
The center image shows one of the small pieces, a hollow cylinder in the center (orange).
The right image is a view from above. The thickness of the shell is dr.

Step 2: Find 'dm', the mass of a representative shell.

$$dm = \rho dV = \rho A dr = \rho(2\pi r h)dr = (2\pi\rho h)rdr \qquad \rho = \frac{M}{V} = \frac{M}{\pi R^2 h}$$

A is the surface area of the cylindrical shell. The length is the circumference and the width is h.

$$dm = (2\pi\rho h)rdr = \left(2\pi\frac{M}{\pi R^2 h}h\right)rdr = \frac{2M}{R^2}rdr$$

Be careful with your variables. Don't confuse R and r. R is the radius of the cylindrical disk. r is the radius of the thin cylindrical shell.

Step 3: Plug in and integrate.

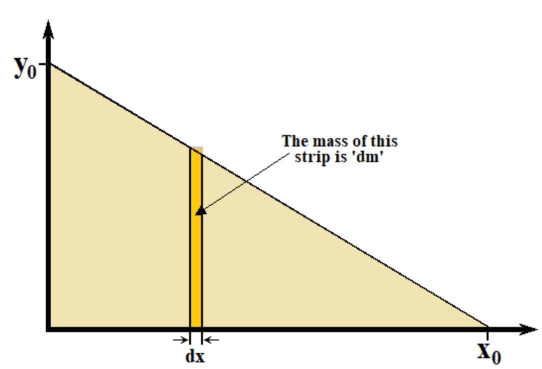
$$I = \int_0^R r^2 dm = \int_0^R r^2\left(\frac{2M}{R^2}rdr\right) = \frac{2M}{R^2}\int_0^R r^3 dr = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \frac{2M}{R^2}\left[\frac{R^4}{4}\right] = \frac{1}{2}MR^2$$

<u>Example</u>: Find the moment of inertia for a flat (right) triangular plate if it is rotating around the y-axis. You may assume uniform thickness and density.

Step 1: Split mass into small pieces, each with the same value of r (as r appears in our equation).

In this case the distance from the axis of rotation (the y-axis), is just x. (r = x)

Step 2: Find 'dm', the mass of the strip.



$$dm = \rho dV = \rho z_0 A = \rho z_0 y dx$$

In this instance, we must find y as a function of x (since y varies with x).

$$y = mx + b \qquad m = \frac{\Delta y}{\Delta x} = \frac{0 - y_0}{x_0 - 0} = -\frac{y_0}{x_0}$$

$$b = y_0 \qquad dm = \rho z_0 y dx = \rho z_0(mx + b)dx$$

We also need the value of ρ. $\rho = \frac{M}{V} = \frac{M}{At} = \frac{M}{\frac{1}{2}bht} = \frac{2M}{bht} = \frac{2M}{x_0 y_0 z_0}$

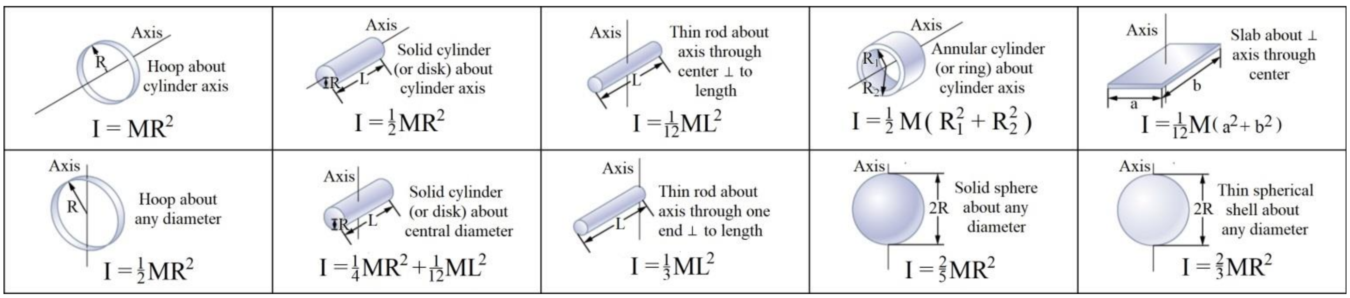
Step 3: Plug in and integrate.

$$I = \int_0^{x_0} r^2 dm = \int_0^{x_0} x^2 dm = \int_0^{x_0} x^2 \rho z_0 (mx + b) dx = \int_0^{x_0} x^2 \frac{2M}{x_0 y_0 z_0} z_0 (mx + b) dx$$

$$I = \frac{2M}{x_0 y_0} \int_0^{x_0} (mx^3 + bx^2) dx = \frac{2M}{x_0 y_0} \left[\frac{1}{4} mx^4 + \frac{1}{3} bx^3\right]_0^{x_0} = \frac{2M}{x_0 y_0} \left[\frac{1}{4} mx_0^4 + \frac{1}{3} bx_0^3\right]$$

$$I = \frac{2M}{x_0 y_0} \left[\frac{1}{4}\left(-\frac{y_0}{x_0}\right) x_0^4 + \frac{1}{3}(y_0) x_0^3\right] = \frac{2M}{x_0 y_0} \left[-\frac{1}{4} y_0 x_0^3 + \frac{1}{3} y_0 x_0^3\right] = \frac{2M}{x_0 y_0} \left[\frac{1}{12} y_0 x_0^3\right] = \frac{1}{6} M x_0^2$$

- Most of the time you will simply pull a formula from a table of standard shapes.



- Moment of inertia is dependent upon the choice of axis.
- If you have multiple objects with the same axis of rotation you simply add their moments of inertia. (Moments of inertia sum)
- The Parallel Axis Theorem allows you to calculate the moment of inertia of an object rotating around an axis that doesn't pass through its center of mass. To do this you need the moment of inertia for an axis parallel to the axis of rotation and passing through the center of mass (I_{CM}) and the separation of the two axes (h).

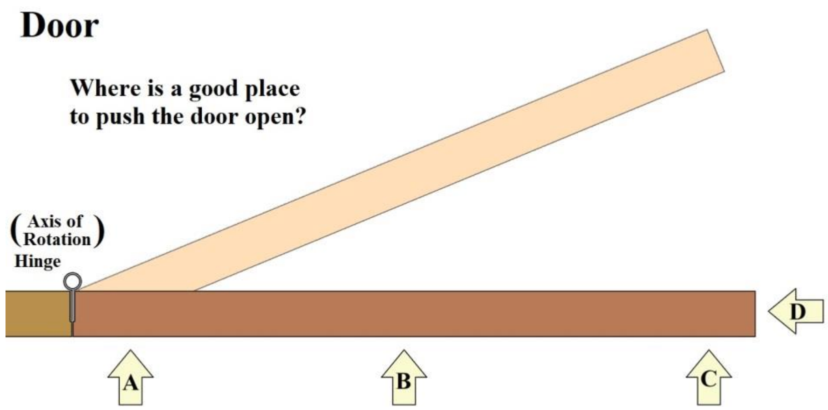
$$I = I_{cm} + Mh^2$$

Example: Find the moment of inertia of a thin rod about axis through one end ⊥ to length via the parallel axis theorem.

$$I = I_{cm} + Mh^2 = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12} ML^2 + \frac{1}{4} ML^2 = \frac{1}{3} ML^2$$

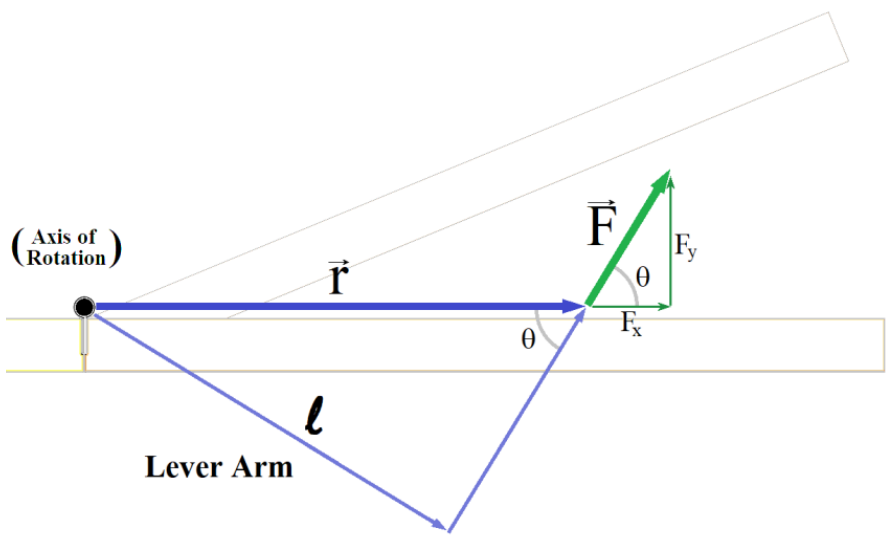
Torque (τ): $\tau = rF_y = rF \sin\theta \qquad \tau = Fl = Fr \sin\theta \qquad |\vec{\tau}| = |\vec{r} \times \vec{F}| = Fr \sin\theta$

Moment of inertia of most objects is fixed (constant). In those cases, Newton's 2nd law ($\vec{\tau} = I\vec{\alpha}$) indicates torque (τ) and angular acceleration (α) are proportional. This allows us to use the behavior of an object (its angular acceleration) to indicate how much torque was delivered when various forces are applied to it. For our example we will use a door.



Force A doesn't work very well. A lot of force leads to little movement of the door.
Force C is the best option. A little force here is usually enough to open the door.
Force B requires more force than C, but not as much as A.
Force D doesn't open the door at all.

- Any force that acts through the axis of rotation generates no torque.
  - To generate torque a force must have a a component $\perp$ to the line connecting the axis of rotation to the point where the force acts.
- The further from the axis of rotation the force is applied the greater the torque.



If we apply a force $\vec{F}$ to an object, we define the vector $\vec{r}$ to start at the axis of rotation and at the position where the force is applied. θ is defined to be the angle between $\vec{F}$ and $\vec{r}$.

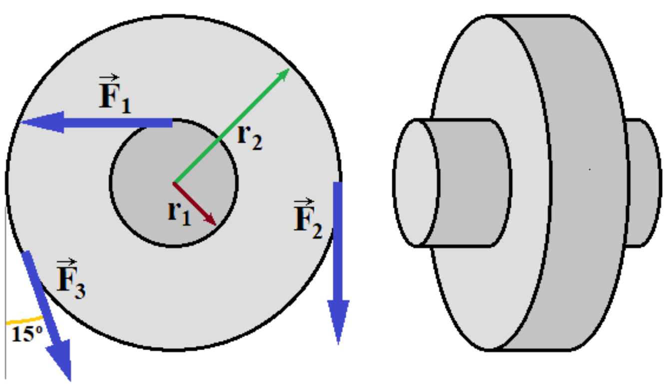
- Breaking $\vec{F}$ into component (parallel and perpendicular to $\vec{r}$) we find:
  - F_x (the component parallel to $\vec{r}$) generates NO torque.
  - F_y (the component perpendicular to $\vec{r}$) generates positive torque as it rotates the door counter clockwise (CCW).

    - The magnitude of the torque ($\tau = |\vec{\tau}|$) is given by: $\tau = rF_y = rF\sin\theta$
  - Forces that create clockwise (CW) rotations are generating negative torque.
  -
- Breaking $\vec{r}$ into component (parallel and perpendicular to $\vec{F}$) we find:
  - The component parallel to $\vec{F}$ has no bearing on the torque at all.
  - The component perpendicular to $\vec{F}$ is called the Lever Arm (l), and it is directly related to the torque. Any increase in the lever arm gives a proportional increase in torque ($\vec{\tau}$).
    - The magnitude of the torque ($\tau = |\vec{\tau}|$) is given by: $\tau = Fl = Fr\sin\theta$
    - We gain "Leverage" by increasing the lever arm.
- Both viewpoints give the same result, which is often represented as a vector cross product.

$$|\vec{\tau}| = \left|\vec{r} \times \vec{F}\right| = Fr\sin\theta$$

There are more advanced methods of calculating vector cross products, but these are rarely used for torque (as we already know the direction along the axis of rotation).

Example: Three forces act on a compound wheel as shown. The forces come from ropes wrapped around the edges of the wheel. The moment of inertia of the wheel is 30.0 kg·m^2 with an inner radius of r_1= 25.0 cm and an outer radius of r_2 = 50.0 cm. Determine the angular acceleration of the wheel in response to the three forces: F_1 = 80.0 N, F_2 = 30.0 N, and F_3 = 20.0 N.



Use Newton's 2nd Law: $\sum\tau = I\alpha$

To find the torque for each force vector:

(1) Determine sign on torque: CCW is '+', CW is '-'

(2) Find the vector $\vec{r}$, note it's magnitude

(3) Determine the angle between $\vec{r}$ and $\vec{F}$ (that's θ)

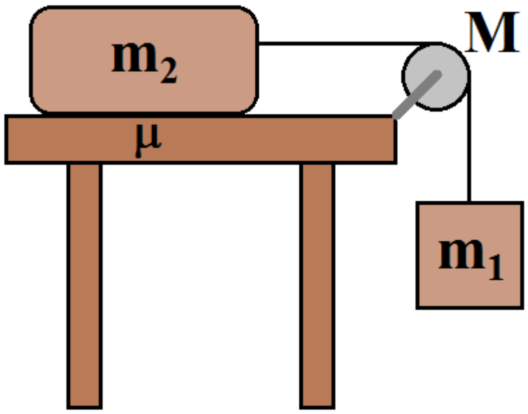
(4) $\tau = rF\sin\theta$

$$\sum\tau = r_1F_1\sin 90° - r_2F_2\sin 90° + r_2F_3\sin 90° = r_1F_1 - r_2F_2 + r_2F_3$$

$$\sum\tau = (0.250\,m)(80.0N) - (0.500\,m)(30.0\,N) + \left((0.500\,m)\right)(20.0\,N)$$

$$\sum\tau = 20.0\,N\cdot m - 15.0\,N\cdot m + 10.0\,N\cdot m = 15.0\,N\cdot m$$

$$\alpha = \frac{\tau_{Net}}{I} = \frac{15.0\,N\cdot m}{30.0\,kg\cdot m^2} = 0.500\frac{rad}{s^2}$$



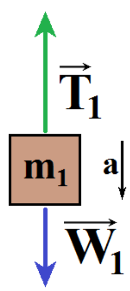
Example: A box of mass m_2 = 10.0 kg sits on a table as shown. On one side a cord connects it to a hanging weight of mass m_1 = 25.0 kg. The cord stretches over a frictionless pulley, a solid disk of uniform density with mass M = 5.00 kg. If the coefficient of kinetic friction between the box and table is 0.300, determine the acceleration of the box on the table.

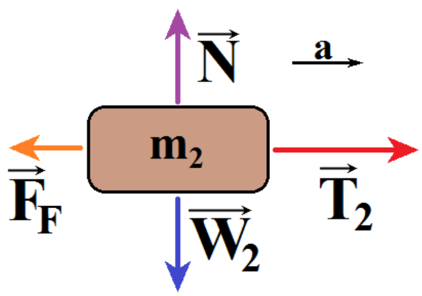
Make 3 force diagrams (one for each object). There are 3 unknowns (2 tensions and acceleration). This means you will need 3 equations, one from each force diagram. Angular acceleration is not another unknown as it is directly related to the acceleration in this problem.

The box on the table (m_2) accelerates to the right, which corresponds to a clockwise (CW) rotation of the pulley (M) and a downward acceleration of the hanging weight (m_1). To match the signs, for this problem we shall let CW rotations and downward accelerations become positive.

$$m_1 g - T_1 = m_1 a$$

$$T_1 = m_1 g - m_1 a$$



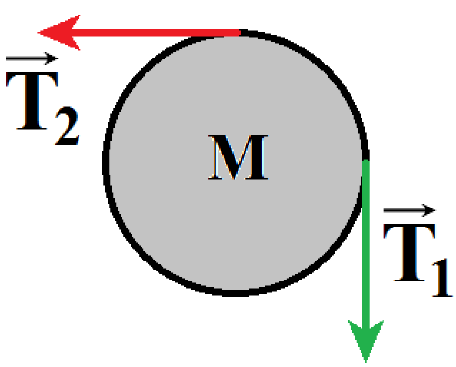


$$N = W_2 = m_2 g$$

$$F_F = \mu_k N = \mu_k m_2 g$$

$$T_2 - F_F = m_2 a$$

$$T_2 = m_2 a + \mu_k m_2 g$$



$$\sum \tau = \tau_1 + \tau_2 = RT_1 - RT_2 = R(T_1 - T_2)$$

$$\sum \tau = R(m_1 g - m_1 a - m_2 a - \mu_k m_2 g)$$

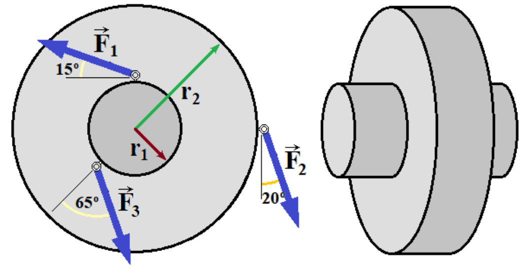
$$I\alpha = \left(\frac{1}{2}MR^2\right)\left(\frac{a}{R}\right) = \frac{1}{2}MRa \qquad I\alpha = \sum \tau$$

$$\frac{1}{2}MRa = R(m_1 g - m_1 a - m_2 a - \mu_k m_2 g)$$

$$\frac{1}{2}Ma = m_1 g - m_1 a - m_2 a - \mu_k m_2 g \qquad m_1 a + m_2 a + \frac{1}{2}Ma = m_1 g - \mu_k m_2 g$$

$$a = \frac{m_1 g - \mu_k m_2 g}{m_1 + m_2 + \frac{1}{2}M} = \frac{(m_1 - \mu_k m_2)g}{m_1 + m_2 + \frac{1}{2}M} = \frac{[(25.0\,kg) - (0.300)(10.0\,kg)](9.80\,m/s^2)}{25.0\,kg + 10.0\,kg + \frac{1}{2}(5.00\,kg)} = 5.75\,\frac{m}{s^2}$$

Example: Three forces act on a compound wheel as shown. The forces come from ropes tied to pins on the edges of the wheel. The moment of inertia of the wheel is 30.0 kg·m^2 with an inner radius of r_1= 25.0 cm and an outer radius of r_2 = 50.0 cm. Determine the angular acceleration of the wheel in response to the three forces: F_1 = 80.0 N, F_2 = 30.0 N, and F_3 = 20.0 N.

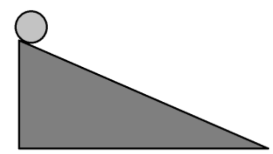


$$\sum \tau = \tau_1 + \tau_2 + \tau_3 = r_1 F_1 \sin 75° - r_2 F_2 \sin 70° + r_1 F_3 \sin 65°$$

$$\tau_{Net} = (0.250\, m)(80.0\, N) \sin 75° - (0.500\, m)(30.0\, N) \sin 70° + (0.250\, m)(20.0\, N) \sin 65°$$

$$\tau_{Net} = 9.7547\, N \cdot m \qquad \alpha = \frac{\tau_{Net}}{I} = \frac{9.7547\, N{\cdot}m}{30.0\, kg{\cdot}m^2} = 0.325 \frac{rad}{s^2}$$

Example: A hoop, a sphere, and a solid cylinder roll down an incline. Each has uniform density, the same mass and radius. If all three are released simultaneously, which gets to the bottom of the incline first?



We shall use $I = cMR^2$ as it applies to all 3 objects with the correct choice of c.
The object with the highest velocity at the bottom gets there first (highest V_{avg})

Conservation of Energy: $E_{init} = E_{Final}$ $\qquad mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

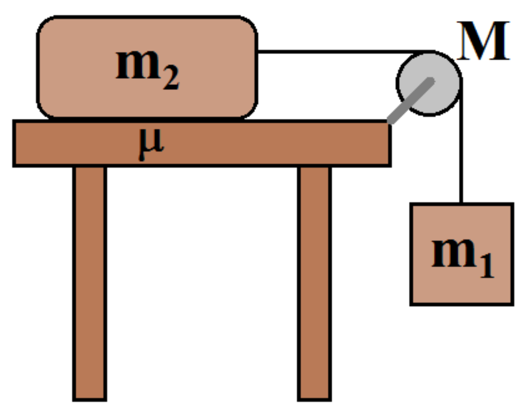
$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}(cMR^2)\left(\frac{v}{R}\right)^2 \qquad Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}cMv^2 = (1+c)\frac{1}{2}Mv^2$$

$$gh = (1+c)\frac{1}{2}v^2 \qquad v^2 = \frac{2gh}{1+c} \qquad v = \sqrt{\frac{2gh}{1+c}}$$

The object with the highest velocity at the bottom has the lowest value of 'c'.

The sphere wins because it has more mass near the axis of rotation.

Example: A box of mass m_2 = 10.0 kg sits on a table as shown. On one side a cord connects it to a hanging weight of mass m_1 = 25.0 kg. The cord stretches over a frictionless pulley, a solid disk of uniform density with mass M = 5.00 kg. The coefficient of kinetic friction between the box and table is 0.300. Determine the velocity of the hanging mass after it has fallen a distance of 0.500 m.



The previous similar problem asked for acceleration, which is related to forces. This problem asks for velocity, which is related to kinetic energy. Using conservation of energy is preferable.

The gravitational potential energy of the box (m_2) and the pulley (M) remain constant. We will ignore these as they will cancel out.

$$E_{init} - E_{Lost} = E_{Final} \qquad E_{init} = m_1 g h_0 \qquad E_{final} = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1 v^2 + \frac{1}{2}I\omega^2 + m_1 g h$$

$$E_{lost} = F_F d = \mu_k N d = \mu_k m_2 g d = \mu_k m_2 g(h_0 - h)$$

$$m_1 g h_0 - \mu_k m_2 g(h_0 - h) = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1 v^2 + \frac{1}{2}I\omega^2 + m_1 g h$$

$$m_1 g h_0 - m_1 g h - \mu_k m_2 g(h_0 - h) = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1 v^2 + \frac{1}{2}I\omega^2$$

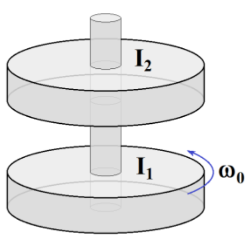
$$m_1 g(h_0 - h) - \mu_k m_2 g(h_0 - h) = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1 v^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2$$

$$(m_1 - \mu_k m_2)g(h_0 - h) = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1 v^2 + \frac{1}{4}Mv^2$$

$$4(m_1 - \mu_k m_2)g(h_0 - h) = 2m_2 v^2 + 2m_1 v^2 + Mv^2 = (2m_2 + 2m_1 + M)v^2$$

$$v^2 = \frac{4(m_1 - \mu_k m_2)g(h_0 - h)}{2m_2 + 2m_1 + M}$$

$$v = \sqrt{\frac{4(m_1 - \mu_k m_2)g(h_0 - h)}{2m_2 + 2m_1 + M}} = \sqrt{\frac{4[25.0\,kg - (0.300)(10.0\,kg)]\left(9.80\frac{m}{s^2}\right)(0.500m)}{2(10.0\,kg) + 2(25.0\,kg) + 5.00\,kg}} = 2.40\frac{m}{s}$$



<u>Example</u>: A solid disk (I_1 = 4.00 kg·m^2) is spinning about a fixed spindle at ω_0 = 15.0 rad/s. A second solid disk (I_2 = 6.00 kg·m^2), which is not rotating, is placed on the spindle and dropped onto the first disk. There is friction between the two disks, and eventually they spin together. Determine (A) the velocity of the two discs once they start spinning together and (B) the energy is lost during the collision.

When spinning objects collide, it's a good indication that conservation of momentum will be relevant.

$$L_{init} = L_{Final} \qquad I_1\omega_0 = (I_1 + I_2)\omega \qquad \omega = \frac{I_1\omega_0}{I_1 + I_2} = \frac{(4.00\,kg{\cdot}m^2)\left(15.0\frac{rad}{s}\right)}{4.00\,kg{\cdot}m^2 + 6.00\,kg{\cdot}m^2} = 6.00\frac{rad}{s}$$

$$E_{Lost} = E_{Init} - E_{Final} = \frac{1}{2}I_1\omega_0^2 - \frac{1}{2}(I_1 + I_2)\omega^2$$

$$E_{Lost} = \frac{1}{2}(4.00\,kg\cdot m^2)\left(15.0\frac{rad}{s}\right)^2 - \frac{1}{2}(4.00\,kg\cdot m^2 + 6.00\,kg\cdot m^2)\left(6.00\frac{rad}{s}\right)^2 = 270\text{ J}$$

<u>Example</u>: An old park has a large turntable for children's play. It is initially at rest with a a radius of 1.20 m and a moment of inertia of 125 kg·m^2. A 50.0 kg woman runs at 8.00 m/s towards the edge of the turntable and jumps on, grabbing hold of the hand rail. Determine the angular velocity of the turntable after she jumps on.

This is also a conservation of angular momentum problem.

$$L_{Init} = L_{woman} = |\vec{r} \times \vec{p}| = |\vec{r} \times m\vec{v}| = Rmv \sin 90° = Rmv$$

$$L_{Final} = (I_{woman} + I_{turntable})\omega = (mR^2 + I)\omega \qquad (mR^2 + I)\omega = Rmv$$

$$\omega = \frac{Rmv}{mR^2 + I} = \frac{(1.20\ m)(50.0\ kg)\left(8.00\frac{m}{s}\right)}{(50.0\ kg)(1.20\ m)^2 + 125\ kg \cdot m^2} = 2.44\frac{rad}{s}$$

<u>Example</u>: The angular momentum of a precision grinding wheel as it starts to rotate is described by $L(t) = L_0(1-e^{-\beta t})$, with L_0 = 315 kg·m^2/s and β =0.247 s^{-1}. Determine the net torque on the wheel at t = 3.17s.

$$\tau_{Net}(t) = \frac{dL}{dt} = \frac{d}{dt}\left[L_0\left(1 - e^{-\beta t}\right)\right] = \frac{d}{dt}\left[L_0 - L_0 e^{-\beta t}\right] = \beta L_0 e^{-\beta t}$$

$$\tau_{Net}(3.17\ s) = (0.247\ \mathrm{s}^{-1})\left(315\ \mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}}\right) e^{-(0.247\ \mathrm{s}^{-1})(3.17\ s)} = 35.6\ N \cdot m$$